# FLOW OF A RAREFIED GAS PAST A SPHERE

PMM Vol. 33, №5, 1969, pp. 895-898

I. N. LARINA
(Moscow)

(Received June 5, 1969)

The flow of a rarefied gas past a sphere is considered. The Crooke equation is solved by the method of integral iterations. An example is used to demonstrate the convergence of the iterative process. The drag of the sphere is calculated; the density, velocity, and temperature distributions in the flow zone are determined.

Let us consider a sphere of radius $R_0$ with the surface temperature $T_w$ in the path of a gas stream of constant velocity $V_\infty$, density $n_\infty$ and temperature $T_\infty$. We attach the coordinate system $xz$ to the sphere as shown in Fig. 1.

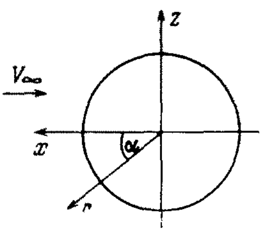


Fig. 1

The Crooke equation for the distribution function $f(\mathbf{x}, \mathbf{u})$ is of the form [1]

$$\mathbf{u}\frac{\partial f}{\partial \mathbf{x}} = \nu[f_0 - f], \qquad f_0 = \frac{n}{\sqrt{(2\pi RT)^3}}\exp\frac{-(\mathbf{u}-\mathbf{v})^2}{2RT} \tag{1}$$

$$n = \int f\, d\mathbf{u}, \qquad n\mathbf{v} = \int \mathbf{u} f\, d\mathbf{u}, \qquad 3nRT = \int (\mathbf{u}-\mathbf{v})^2 f\, d\mathbf{u}$$

$$\nu = \frac{16}{5}\frac{nRT}{n_\infty\lambda_\infty\sqrt{2\pi RT_\infty}}\left(\frac{T}{T_\infty}\right)^{-\omega}$$

($\omega$ is a constant for the given gas)

We shall think of a gas as consisting of solid spherical molecules for which $\omega = 0.5$.

We stipulate that the distribution function of the molecules travelling towards the sphere satisfies the following condition at infinity:

$$f_\infty = \frac{n_\infty}{\sqrt{(2\pi RT_\infty)^3}}\exp\frac{-(\mathbf{u}-\mathbf{V}_\infty)^2}{2RT_\infty} \tag{2}$$

We assume diffuse reflection of the molecules at the surface of the sphere with the distribution function

$$f_w = \frac{n_w}{\sqrt{(2\pi RT_w)^3}}\exp\frac{-\mathbf{u}^2}{2RT_w} \tag{3}$$

The density $n_w$ of the molecules reflected from the body is not known in advance. It can be determined from the condition of nonleakage of the mass of the surface of the body,

$$n_w = -\left(\frac{2\pi}{RT_w}\right)^{1/2}\int \mathbf{u}\cdot\mathbf{n} f\, d\mathbf{u} \tag{4}$$

where the integration must be carried out over the velocities of the particles striking the body. Let us introduce the dimensionless quantities

$$\mathbf{u} = \mathbf{u}^* v_\infty, \qquad \mathbf{x} = 2R_0\mathbf{x}^*, \qquad f = n_\infty(2RT_\infty)^{-3/2} f^*$$

$$n = n^* n_\infty, \qquad \mathbf{V}_\infty = \mathbf{s} v_\infty, \qquad T = T^* T_\infty, \qquad T_w = T_w^* T_\infty, \qquad \nu = \nu^* v_\infty$$

taking $v_\infty = \sqrt{2RT_\infty}$ as the characteristic velocity and the diameter of the sphere as the characteristic length.

Next, let us rewrite Eq. (1) in these dimensionless parameters (omitting the asterisks),

$$\mathbf{u}\frac{\partial f}{\partial \mathbf{x}} = \nu[f_0 - f] \tag{5}$$

$$f_0 = \frac{n}{\sqrt{(\pi T)^3}} \exp \frac{-(\mathbf{u}-\mathbf{v})^2}{T}, \qquad \nu = \frac{16n\sqrt{T}}{5\sqrt{\pi}K}$$

$$n = \int f\, d\mathbf{u}, \qquad n\mathbf{v} = \int \mathbf{u} f\, d\mathbf{u}, \qquad nT = {}^2/_3 \int (\mathbf{u}-\mathbf{v})^2 f d\mathbf{u} \tag{6}$$

Here $K$ is the Knudsen number. Condition (2) can be rewritten as

$$f_\infty = \pi^{-3/2} \exp\left[-(\mathbf{u}-\mathbf{s})^2\right] \tag{7}$$

and conditions (3) and (4) for the particles reflected by the body as

$$f_w = \frac{n_w}{\sqrt{\pi^3 T_w^{\,3}}} \exp \frac{-\mathbf{u}^2}{T_w}, \qquad n_w = -2\left(\frac{\pi}{T_w}\right)^{1/2} \int \mathbf{u}\cdot\mathbf{n} f\, d\mathbf{u} \tag{8}$$

(integration is over the velocities of the particles inc.:dent on the body).

Thus, solution of the problem of flow past a sphere reduces to the solution of Eq. (5) under boundary conditions (7), (8).

Let us rewrite Eq. (5) in integral form,

$$f(\mathbf{x}, \mathbf{u}) = f(\mathbf{x} - \mathbf{a}y_s) \exp\left[-\frac{1}{u}\int_0^{y_s} \nu(\mathbf{x} - \mathbf{a}y')\, dy'\right] +$$

$$+\frac{1}{u}\int_0^{y_s} \nu(\mathbf{x} - \mathbf{a}y) f_0(\mathbf{x} - \mathbf{a}y) \exp\left[-\frac{1}{u}\int_0^{y} \nu(\mathbf{x} - \mathbf{a}y')\, dy'\right] dy \tag{9}$$

where $\mathbf{a}$ is the unit vector which defines the direction of the particle velocity $\mathbf{u}$.

We convert to the polar coordinates $r$, $\alpha$ (Fig. 1) in the coordinate plane $xz$, and to the spherical coordinate system $u$, $\vartheta$, $\varphi$ in the velocity space, taking the ray $\alpha$ as our axis $\vartheta = 0$ at each point $r$, $\alpha$ of the field. (We measure the angle $\varphi$ from the $z$-axis). We denote the solid angle at which the sphere is seen from the point $r$ by $\Omega(r)$. This enables us to write the following expression for the quantity $y_s$ occurring in Eq. (9):

$$y_s = \infty, \quad \mathbf{u} \notin \Omega(r)$$

$$y_s = \sqrt{0.25 + r^2 - 2r\cos\beta}, \qquad \mathbf{u} \in \Omega(r)$$

$$(\beta = \arcsin(r\sin\vartheta) - \vartheta)$$

Equations (9), (6) form a nonlinear system of equations with the unknowns $n$, $\mathbf{v}$, $T$, $f$ which we can solve by the method of iterations. The latter consists in substituting the known values of $n^{n-1}$, $\mathbf{v}^{n-1}$, $T^{n-1}$ and the value of $n_w^{n-1}$ obtained from formula (8) for a known $f^{n-1}$ into the right side of Eq. (9) and then using the resulting value of the distribution function $f^n$ to find $n^n$, $\mathbf{v}^n$, $T^n$ and $n_w^n$ in the next approximation. This iteration procedure is convenient because it does not require memorization of the distribution function. In each iteration we need memorize only the first four moments of the distribution function and the value of the reflected-particle density $n_w$ at the body for each point of the coordinate grid.

This reduces solution of the problem to the computation of a quadruple integral at each point of the field grid. Integration along the ray with the direction $\mathbf{a}$ in computing the distribution function $f(r, \alpha, \mathbf{u})$ can be carried out by the trapezoid method.

The interval $h$ of integration over $y$ must be chosen [2] on the basis of $u$ and $\nu$, since the exponent in the integrand decays rapidly for small $u$ and large $\nu$.

Taking the step $h$ from the point $r, \alpha$ $(y=0)$ along the ray coincident with the vector $\mathbf{a}(\vartheta, \varphi)$, we arrive at the point with the coordinates $r'$, $\alpha'$,

$$r' = \sqrt{h^2 + r^2 - 2rh\cos\vartheta}, \qquad \cos\alpha' = (r\cos\alpha + h\cos\chi)/r'$$

$$(\cos\chi = \sin\alpha\,\sin\vartheta\,\cos\varphi - \cos\varphi\,\cos\vartheta)$$

The angle $\alpha'$ is the angle between the $x$-axis and the vector $\mathbf{x} - \mathbf{a}h$.

The axial symmetry of the problem enables us to choose the values of $n^{n-1}$, $v_x^{n-1}$, $v_z^{n-1}$, $T^{n-1}$ for the integrand at the point $r'$, $\alpha'$ from the totality of values of $n, v_x, v_z, T$ in the plane $xz$. Linear interpolation can be used in choosing these values from tables.

The triple integrals over the velocities involved in computing $n$, $v_x, v_z$, $T$ can be calculated by the Monte-Carlo method. Accuracy of computation can be enhanced by subtracting from the integrand the locally Maxwellian distribution function multiplied by 1, $u_x$, $u_z$, $(\mathbf{u} - \mathbf{v})^2$, respectively, with the parameters $n^{n-1}$, $v_x^{n-1}$, $v_z^{n-1}$, $T^{n-1}$ taken from the preceding iteration.

We carried out our computations on a BESM-6 computer for the following values of the stream parameters: $K = 1$, $s = 1$ $(M = 1.095)$ and $T_w = T_\infty$. The unperturbed-stream conditions were specified for $r = 5\lambda_\infty$. The grid in the plane $r, \alpha$ was chosen as follows: the interval $\Delta\alpha$ was set equal to 0.125 ; the interval in $r$ was $\Delta r = 0.0625\lambda_\infty$ up to $r = 1.5\lambda_\infty$; for $r > 1.5\lambda_\infty$ we chose the interval $\Delta r = 0.25\lambda_\infty$.

The integrals over the velocities were computed for 1000 random points. Because of reduced dispersion the density $n$ and the velocities $v_x, v_z$ were computed to within ~2% and the temperature $T$ to within ~3%. (For $K = 0.5$ this degree of accuracy was attainable with 500 drawings).

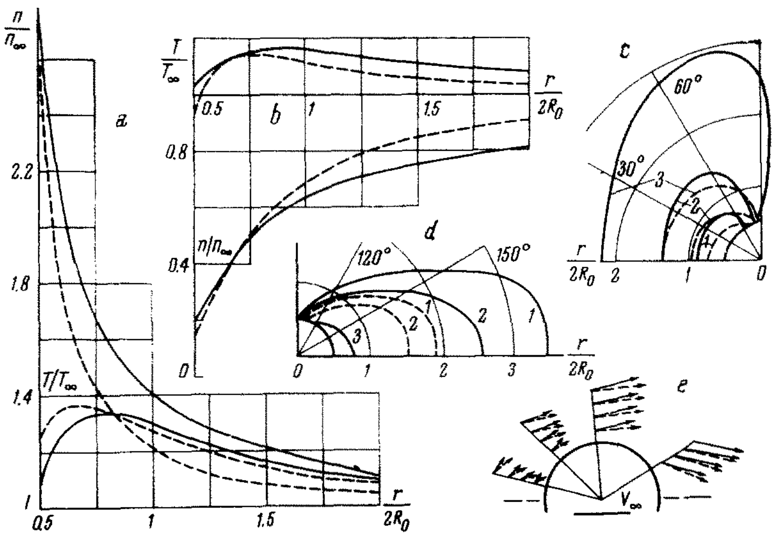


Fig. 2

The density $n$, the velocities $v_x$, $v_z$, the temperature $T$, and the density $n_w$ of the particles reflected from the body were set equal to their values in free-molecular flow for the first iteration. The computations indicated slow convergence of the integral iterations. Arrival at a solution which did not vary in subsequent iterations required 14 iterations. Eight iterations were sufficient throughout the domain behind the sphere and within $r < 0.5\lambda_\infty$ in front of the sphere. However, the density at a distance equal to the free path length in front of the sphere was 1.5 times larger after 14 iterations than it was after eight iterations.

Thus, the familiar slow convergence of integral iterations typical of one-dimensional problems also applies in the case of three-dimensional problems.

The distribution curves of the density $n / n_\infty$ and temperature $T / T_\infty$ along the axis of symmetry in front of the body appear in Fig. 2a; the corresponding distributions behind the body are shown in Fig. 2b. Here and in the remaining figure the broken curves apply to free-molecular flow past the body. A shock wave does not arise for the stream parameters considered. Collisions between particles in the stream have the effect of markedly increasing their density behind the body and decreasing it in front of the body. The stream becomes unperturbed further away from the body than in the case of free-molecular flow. This picture is valid for all directions $\alpha$. We see this from Figs. 2c and 2d which show the lines of equal values of the density $n / n_\infty$. The number 1 in Fig. 2c, denotes the curves for the values $n / n_\infty = 1.5$, the number 2 the curves for $n / n_\infty = 1.25$, and the number 3 the curves for $n / n_\infty = 1.1$. For $K = 1$ the stream in front of the body not only becomes more dense; it also decelerates more rapidly, as we see from Fig. 2e, which shows the vector velocity field around the sphere (and indicates the scale on which the velocities are plotted).

It is interesting to note that the gas density at the distance $r > 0.5\,\lambda_\infty$ behind the sphere (Fig. 2d) is lower than that for a free-molecular stream (the number 1 identifies the curve for $n / n_\infty = 0.9$, the number 2 the curve for $n / n_\infty = 0.85$, the number 3 the curve for $n / n_\infty = 0.5$).

The drag of the sphere was computed at each iteration. The third iteration yielded the value $C_x = 3.55$, for the drag coefficient; the values in subsequent iterations fluctuated within the error bracket ~3% of the drag computations. (For free-molecular flow past a sphere with $M = 1.095$ the drag coefficient turned out to be $C_x = 4.55$.)

The first results just presented indicate that the procedure described can be used for calculations over quite broad ranges of Mach and Knudsen numbers.

## BIBLIOGRAPHY

1. Bhatnager P. L., Gross, E. P. and Krook, M., Model of Collision Processes in Gases, I. Small-amplitude processes in charged and neutral one-component systems. Phys. Rev. Vol. 94, №3, 1954.
2. Cheremisin, F. G., The structure of the shock wave in a gas consisting of ideally elastic rigid spherical molecules. In: Numerical Methods in the Theory of Rarefied Gases. Moscow, Izd. VTs Akad. Nauk SSSR, 1969.

Translated by A. Y.